Motivic Galois theory for motives of niveau ≤ 1

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Abstract. Let $\mathcal{T}$ be a Tannakian category over a field k of characteristic 0 and $\pi(\mathcal{T})$ its fundamental group. In this paper we prove that there is a bijection between the $\otimes$-equivalence classes of Tannakian subcategories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$.
We apply this result to the Tannakian category $\mathcal{T}_1(k)$ generated by motives of niveau ≤ 1 defined over k, whose fundamental group is called the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of motives of niveau ≤ 1. We find four short exact sequences of affine group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$, correlated one to each other by inclusions and projections. Moreover, given a 1-motive M, we compute explicitly the biggest Tannakian subcategory of the one generated by M, whose fundamental group is commutative.

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Introduction.

Let k be a field of characteristic 0 and $\overline{k}$ its algebraic closure. Let $\mathcal{T}$ be a Tannakian category over k. The tensor product of $\mathcal{T}$ allows us to define the notion of Hopf algebras in the category Ind$\mathcal{T}$ of Ind-objects of $\mathcal{T}$. The category of affine group $\mathcal{T}$-schemes is the opposite of the category of Hopf algebras in Ind$\mathcal{T}$.
The fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$ is the affine group $\mathcal{T}$-scheme $\mathrm{Sp}(\Lambda)$, whose Hopf algebra Λ satisfies the following universal property: for each object X of $\mathcal{T}$, there exists a morphism $X \longrightarrow \Lambda \otimes X$ functorial in X. Those morphisms $\{X \longrightarrow \Lambda \otimes X\}_{X \in \mathcal{T}}$ define *an action of the fundamental group* $\pi(\mathcal{T})$ *on each object of* $\mathcal{T}$.

For each Tannakian subcategory $\mathcal{T}'$ of $\mathcal{T}$, let $H_{\mathcal{T}}(\mathcal{T}')$ be the kernel of the faithfully flat morphism of affine group $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion functor $i : \mathcal{T}' \longrightarrow \mathcal{T}$. In particular we have the short exact sequence of affine group $\pi(\mathcal{T})$-schemes

$$0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow \pi(\mathcal{T}') \longrightarrow 0. \tag{0.1}$$

In [6] 6.6, Deligne proves that the Tannakian category $\mathcal{T}'$ is equivalent, as tensor category, to the subcategory of $\mathcal{T}$ generated by those object on which the action of $\pi(\mathcal{T})$ induces a trivial action of $H_{\mathcal{T}}(\mathcal{T}')$. In particular, this implies that the fundamental group of $\pi(\mathcal{T}')$ of $\mathcal{T}'$ is isomorphic to the affine group $\mathcal{T}$-scheme $\pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}')$. The affine group $\pi(\mathcal{T})$-scheme $H_{\mathcal{T}}(\mathcal{T}')$ characterizes the Tannakian subcategory $\mathcal{T}'$ modulo $\otimes$-equivalence. In fact we have the following result (theorem 1.7): *there is bijection between the $\otimes$-equivalence classes of Tannakian subcategories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$, which associates*

- to each Tannakian subcategory $\mathcal{T}'$ of $\mathcal{T}$, the kernel $H_{\mathcal{T}}(\mathcal{T}')$ of the morphism of $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion $i : \mathcal{T}' \longrightarrow \mathcal{T}$;

- to each normal affine group sub-$\mathcal{T}$-schemes H of $\pi(\mathcal{T})$, the Tannakian subcategory $\mathcal{T}(H)$ of $\mathcal{T}$, whose fundamental group $\pi(\mathcal{T}(H))$ is the affine group $\mathcal{T}$-scheme $\pi(\mathcal{T})/H$.

Hence we obtain a clear dictionary between Tannakian subcategories of $\mathcal{T}$ and normal affine group sub-$\mathcal{T}$-schemes of the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$. Pierre Deligne has pointed out to the author that we can see this bijection as a "reformulation" of [11] II 4.3.2 b) and g). The proof of theorem 1.7 is based on [7] theorem 8.17.

We apply this dictionary to the Tannakian category $\mathcal{T}_1(k)$ generated by motives of niveau ≤ 1 defined over k (in an appropriate category of mixed realizations). We want to precise that in this article we restrict to motives of niveau ≤ 1 because we are interested in motivic (and hence geometric) results and until now we know concretely only motives of niveau ≤ 1. The fundamental group of $\mathcal{T}_1(k)$ is called the *motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of motives of niveau ≤ 1.*

For each fibre functor ω of $\mathcal{T}_1(k)$ over a k-scheme S, $\omega\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ is the affine group S-scheme $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$ which represents the functor which associates to each S-scheme T, $u : T \longrightarrow S$, the group of automorphisms of $\otimes$-functors of the functor $u^*\omega$. In particular, for each embedding $\sigma : k \longrightarrow \mathbb{C}$, the fibre functor ω_σ "Hodge realization" furnishes the algebraic group $\mathbb{Q}$-scheme

$$\omega_\sigma \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) = \mathrm{Spec}\left(\omega_\sigma(\Lambda)\right) = \underline{\mathrm{Aut}}_{\mathbb{Q}}^{\otimes}(\omega_\sigma) \tag{0.2}$$

which is the *Hodge realization of the motivic Galois group of $\mathcal{T}_1(k)$.*

The weight filtration W_* of the motives of niveau ≤ 1 induces an increasing filtration W_* of 3 steps in the affine group $\mathcal{T}_1(k)$-scheme $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. Each of these 3 steps can be defined as intersection of normal affine group sub-$\mathcal{T}_1(k)$-schemes of

$\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$, which correspond to determined Tannakian sub-categories of $\mathcal{T}_1(k)$. For the Tannakian subcategories $\mathrm{Gr}_0^W \mathcal{T}_1(k), \mathrm{Gr}_{-2}^W \mathcal{T}_1(k), \mathrm{Gr}_*^W \mathcal{T}_1(k)$ and $W_{-1}\mathcal{T}_1(k)$ of $\mathcal{T}_1(k)$, we compute the corresponding normal affine group sub-$\mathcal{T}_1(k)$-schemes which will furnish us, according to (0.1), four exact short sequences of affine group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ (theorem 3.4). These short exact sequences are correlated one to each other by inclusions and projections, and they involve also the filtration W_* of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. One of these short exact sequences is

$$(0.3) \qquad 0 \longrightarrow \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{\pi} \mathrm{Gal}(\overline{k}/k) \longrightarrow 0.$$

If τ is an element of $\mathrm{Gal}(\overline{k}/k)$, we have that $\pi^{-1}(\tau)$ is $\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, where Id and $\tau \circ \mathrm{Id}$ have to be regarded as functors on $\mathcal{T}_1(\overline{k})$ (corollary 3.6). By (0.2), the short exact sequence (0.3) is the motivic version of the short exact sequence of $\mathbb{Q}$-algebraic groups

$$(0.4) \qquad 0 \longrightarrow \underline{\mathrm{Aut}}_{\mathbb{Q}}^{\otimes}(\omega_{\overline{\sigma}\,|\mathcal{T}_1(\overline{k})}) \longrightarrow \underline{\mathrm{Aut}}_{\mathbb{Q}}^{\otimes}(\omega_{\sigma}) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 0$$

where $\overline{\sigma} : \overline{k} \longrightarrow \mathbb{C}$ is the embedding of $\overline{k}$ in $\mathbb{C}$ which extends $\sigma : k \longrightarrow \mathbb{C}$. This last sequence is the restriction to motives of niveau ≤ 1 of the sequence found by P. Deligne and U. Jannsen in [5] II 6.23 and [8] 4.7 respectively. Moreover the equality $\pi^{-1}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$ is the motivic version of the one found by P. Deligne and U. Jannsen (loc. cit.).

Let M be a 1-motive defined over k. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(M)$ of M is the fundamental group of the Tannakian subcategory $\langle M \rangle^{\otimes}$ of $\mathcal{T}_1(k)$ generated by M. In [2], we compute the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$: it is the semi-abelian variety defined by the adjoint action of the graduated $\mathrm{Gr}_*^W(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ on itself. This result allows us to compute the derived group of the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ (proposition 4.4). Applying then theorem 1.7, we construct explicitly the biggest Tannakian subcategory of $\langle M \rangle^{\otimes}$ which has a commutative motivic Galois group (theorem 4.9).

The "Hodge realization" of the motivic Galois group of motives was partially studied by P. Deligne and U. Jannsen in [5] II §6 and [8] 4.6 respectively. They don't restrict to motives of niveau ≤ 1 and hence we find the motivic version of the restriction to motives of niveau ≤ 1 of some of theirs results (corollary 3.6).

The dictionary between Tannakian subcategories of a Tannakian category $\mathcal{T}$ and normal affine group sub-$\mathcal{T}$-schemes of the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$ has applications also in a "non-motivic" context. It can be apply whenever one has objects generating a Tannakian category.

In the first section, we recall the definition and some properties of the fundamental group $\pi(\mathcal{T})$ of a Tannakian category $\mathcal{T}$ and we prove the bijection between the $\otimes$-equivalence classes of Tannakian subcategories of $\mathcal{T}$ and normal affine sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$.

In the section 2, we apply the definitions of the first section to the Tannakian category $\mathcal{T}_1(k)$ generated by motives of niveau ≤ 1 defined over k (in an appropriate category of mixed realizations): the motivic Galois group of a motive is the fundamental group of the Tannakian category generated by this motive. We end this section giving several examples of motivic Galois groups.

In the third section, we apply the dictionary between Tannakian sub-categories and normal affine group sub-$\mathcal{T}_1(k)$-schemes to some Tannakian subcategories of $\mathcal{T}_1(k)$. In particular, we get the motivic version of [5] II 6.23 (a), (c) and [8] 4.7 (c), (e).

In section 4, we compute the biggest Tannakian subcategory of the Tannakian category generated by a 1-motive, which has a commutative motivic Galois group.

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In this paper k is a field of characteristic 0 embeddable in $\mathbb{C}$ and $\overline{k}$ its algebraic closure.

1. Motivic Galois theory.

1.1. Let $\mathcal{T}$ be a Tannakian category over k, i.e. a tensor category over k with a fibre functor over a nonempty k-scheme. A **Tannakian subcategory of $\mathcal{T}$** is a strictly full subcategory $\mathcal{T}'$ of $\mathcal{T}$ which is closed under the formation of subquotients, direct sums, tensor products and duals, and which is endowed with the restriction to $\mathcal{T}'$ of the fibre functor of $\mathcal{T}$.

The tensor product of $\mathcal{T}$ extends to a tensor product in the category $\mathrm{Ind}\mathcal{T}$ of Ind-objects of $\mathcal{T}$. A **commutative ring** in $\mathrm{Ind}\mathcal{T}$ is an object A of $\mathrm{Ind}\mathcal{T}$ together with a commutative associative product $A \otimes A \longrightarrow A$ admitting an identity $1_{\mathcal{T}} \longrightarrow A$, where $1_{\mathcal{T}}$ is the unit object of $\mathcal{T}$. In $\mathrm{Ind}\mathcal{T}$ we can define in the usual way the notion of morphisms of commutative rings, the notion of A-module (an object of $\mathrm{Ind}\mathcal{T}$ endowed with a structure of A-module), ... The **category of affine $\mathcal{T}$-schemes** is the opposite of the category of commutative rings in $\mathrm{Ind}\mathcal{T}$. We denote $\mathrm{Sp}(A)$ the affine $\mathcal{T}$-scheme defined by the ring A. The final object of this category is the affine $\mathcal{T}$-scheme $\mathrm{Sp}(1_{\mathcal{T}})$ defined by the ring $1_{\mathcal{T}}$. A module over $\mathrm{Sp}(A)$ is an A-module and for each morphism $\mathrm{Sp}(B) \longrightarrow \mathrm{Sp}(A)$, the functor $M \longmapsto B \otimes_A M$ is called "the inverse image over $\mathrm{Sp}(B)$." An **affine group $\mathcal{T}$-scheme** is a group object in the category of affine $\mathcal{T}$-schemes, i.e. $\mathrm{Sp}(A)$ with A endowed with a structure of Hopf algebra. An **action** of an affine group $\mathcal{T}$-scheme $\mathrm{Sp}(A)$ on an object X of $\mathcal{T}$ is a morphism $X \longrightarrow X \otimes A$ satisfying the usual axioms for a A-comodule. The **Lie algebra** of an affine group $\mathcal{T}$-scheme is a pro-object L of $\mathcal{T}$ endowed with a Lie algebra structure.

1.2. REMARKS:

(1) Since $\mathrm{End}(1_{\mathcal{T}}) = k$, the Tannakian subcategory of $\mathcal{T}$ consisting of direct sum of copies of $1_{\mathcal{T}}$ is equivalent to the Tannakian category of finite-dimensional k-vector spaces. In particular, *every Tannakian category over k contains, as Tannakian subcategory, the category of finite-dimensional k-vector spaces* $\mathrm{Vec}(k)$. Considering Ind-objects, we have that *each affine k-scheme defines an affine $\mathcal{T}$-scheme.* In particular, $\mathrm{Spec}(k)$ corresponds to $\mathrm{Sp}(1_{\mathcal{T}})$ (cf. [6] 5.6).

(2) Let $\mathcal{T}$ be the Tannakian category of representations of an affine group scheme G over k: $\mathcal{T} = \mathrm{Rep}_k(G)$. In this case, affine $\mathcal{T}$-schemes are affine k-schemes endowed with an action of G. The inclusion of affine k-schemes in the category of affine $\mathcal{T}$-schemes described in 1.2 (1), is realized adding the trivial action of G (cf. [6] 5.8).

1.3. The **fundamental group** $\pi(\mathcal{T})$ of a Tannakian category $\mathcal{T}$ is the affine group $\mathcal{T}$-scheme $\mathrm{Sp}(\Lambda)$, where Λ is the Hopf algebra of $\mathrm{Ind}\mathcal{T}$ satisfying the following universal property: for each object X of $\mathcal{T}$, there exists a morphism

$$\lambda_X : X^{\vee} \otimes X \longrightarrow \Lambda \tag{1.3.1}$$

functorial in X, i.e. for each $f : X \longrightarrow Y$ in $\mathcal{T}$ the diagram

$$\begin{array}{ccc} Y^{\vee} \otimes X & \stackrel{f^t \otimes 1}{\longrightarrow} & X^{\vee} \otimes X \\ {\scriptstyle 1\otimes f} \downarrow & & \downarrow {\scriptstyle \lambda_X} \\ Y^{\vee} \otimes Y & \stackrel{\lambda_Y}{\longrightarrow} & \Lambda \end{array}$$

is commutative. The universal property of Λ reads: for each Ind-object U of $\mathcal{T}$ the application

$$\begin{aligned} \mathrm{Hom}(\Lambda, U) &\longrightarrow \{u_X : X^{\vee} \otimes X \longrightarrow U, \quad \text{functorial in } X\} \\ f &\longmapsto f \circ \lambda_X \end{aligned} \tag{1.3.2}$$

is a bijection. The existence of the fundamental group $\pi(\mathcal{T})$ is proved in [7] 8.4, 8.10, 8.11 (iii).

The morphisms (1.3.1), which can be rewritten on the form

$$X \longrightarrow X \otimes \Lambda, \tag{1.3.3}$$

define *an action of the fundamental group $\pi(\mathcal{T})$ on each object of $\mathcal{T}$.* In particular, the morphism $\Lambda \longrightarrow \Lambda \otimes \Lambda$ represents the action of $\pi(\mathcal{T})$ on itself by inner automorphisms (cf. [6] 6.1).

1.4. EXAMPLES:

(1) Let $\mathcal{T} = \mathrm{Vec}(k)$ be the Tannakian category of finite dimensional k-vector spaces. From the main theorem of Tannakian category, we know that $\mathrm{Vec}(k)$ is

equivalent to the category of finite-dimensional k-representations of Spec (k). In this case, affine $\mathcal{T}$-schemes are affine k-schemes and $\pi(\mathrm{Vec}(k))$ is Spec (k).

(2) Let $\mathcal{T}$ be the Tannakian category of k-representations of an affine group scheme G over k: $\mathcal{T} = \mathrm{Rep}_k(G)$. From [6] 6.3, the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$ is the affine group k-scheme G endowed with its action on itself by inner automorphisms.

1.5. From [7] 6.4, to any exact and k-linear $\otimes$-functor $u : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between Tannakian categories over k, corresponds a morphism of affine group $\mathcal{T}_2$-schemes

$$U : \pi(\mathcal{T}_2) \longrightarrow u\pi(\mathcal{T}_1). \tag{1.5.1}$$

For each object X_1 of $\mathcal{T}_1$, the action (1.3.3) of $\pi(\mathcal{T}_1)$ on X_1 induces an action of $u\pi(\mathcal{T}_1)$ on $u(X_1)$. Via (1.5.1) this last action induces the action (1.3.3) of $\pi(\mathcal{T}_2)$ on the object $u(X_1)$ of $\mathcal{T}_2$.

By the theorem [11] II 4.3.2 (g) we have the following dictionary between the functor u and the morphism U:

(1) U is faithfully flat (i.e. flat and surjective) if and only if u is fully faithful and every subobject of $u(X_1)$ for X_1 an object of $\mathcal{T}_1$, is isomorphic to the image of a subobject of X_1.

(2) U is a closed immersion if and only if every object of $\mathcal{T}_2$ is isomorphic to a subquotient of an object of the form $u(X_1)$, for X_1 an object of $\mathcal{T}_1$.

1.6. We can now state the dictionary between Tannakian subcategories of $\mathcal{T}$ and normal affine group sub-$\mathcal{T}$-schemes of the fundamental group $\pi(\mathcal{T})$ of $\mathcal{T}$:

1.7. **Theorem**

Let $\mathcal{T}$ be a Tannakian category over k, with fundamental group $\pi(\mathcal{T})$. There is a bijection between the $\otimes$-equivalence classes of Tannakian subcategories of $\mathcal{T}$ and the normal affine group sub-$\mathcal{T}$-schemes of $\pi(\mathcal{T})$:

$$\begin{array}{rcl} \begin{pmatrix} \otimes - \text{equiv. classes of} \\ \text{Tannakian subcat. of } \mathcal{T} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \text{normal affine group} \\ \text{sub} - \mathcal{T} - \text{schemes of } \pi(\mathcal{T}) \end{pmatrix} \\ \mathcal{T}' & \longrightarrow & H_{\mathcal{T}}(\mathcal{T}') = \ker\left(\pi(\mathcal{T}) \xrightarrow{I} i\pi(\mathcal{T}')\right) \\ \mathcal{T}(H) \text{ s.t. } \pi(\mathcal{T}(H)) \cong \pi(\mathcal{T})/H & \longleftarrow & H \end{array}$$

which associates

- to each Tannakian subcategory $\mathcal{T}'$ of $\mathcal{T}$, the kernel $H_{\mathcal{T}}(\mathcal{T}')$ of the morphism of affine group $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$ corresponding to the inclusion functor $i : \mathcal{T}' \longrightarrow \mathcal{T}$. In particular, we can identify the fundamental group $\pi(\mathcal{T}')$ of $\mathcal{T}'$ with the affine group $\mathcal{T}$-scheme $\pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}')$.

- to each normal affine group sub-$\mathcal{T}$-scheme H of $\pi(\mathcal{T})$, the Tannakian subcategory $\mathcal{T}(H)$ of $\mathcal{T}$ whose fundamental group $\pi(\mathcal{T}(H))$ is the affine group $\mathcal{T}$-scheme

$\pi(\mathcal{T})/H$. *Explicitly,* $\mathcal{T}(H)$ *is the Tannakian subcategory of* $\mathcal{T}$ *generated by the objects of* $\mathcal{T}$ *on which the action (1.3.3) of* $\pi(\mathcal{T})$ *induces a trivial action of* H.

PROOF: Let $i : \mathcal{T}' \longrightarrow \mathcal{T}$ be the inclusion functor of a Tannakian sub-category $\mathcal{T}'$ of $\mathcal{T}$. From 1.5 to this functor corresponds a faithfully flat morphism of affine group $\mathcal{T}$-schemes $I : \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}')$. Denote by

$$H_{\mathcal{T}}(\mathcal{T}') = \ker\left(\pi(\mathcal{T}) \xrightarrow{I} i\pi(\mathcal{T}')\right)$$

the affine group $\mathcal{T}$-scheme which is the kernel of the morphism I. In particular, we have the exact sequence of affine group $\mathcal{T}$-schemes

$$0 \longrightarrow H_{\mathcal{T}}(\mathcal{T}') \longrightarrow \pi(\mathcal{T}) \longrightarrow i\pi(\mathcal{T}') \longrightarrow 0.$$

According to theorem 8.17 of [7], the inclusion functor $i : \mathcal{T}' \longrightarrow \mathcal{T}$ identifies $\mathcal{T}'$ with the Tannakian subcategory of objects of $\mathcal{T}$ on which the action (1.3.3) of $\pi(\mathcal{T})$ induces a trivial action of $H_{\mathcal{T}}(\mathcal{T}')$. In particular, we get that

$$\pi(\mathcal{T}') \cong \pi(\mathcal{T})/H_{\mathcal{T}}(\mathcal{T}'). \tag{1.7.1}$$

We now check the injectivity: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two Tannakian sub-categories of $\mathcal{T}$ such that $H_{\mathcal{T}}(\mathcal{T}_1) = H_{\mathcal{T}}(\mathcal{T}_2)$, then by (1.7.1) they have also the same fundamental group. But this means that there is a $\otimes$-equivalence of categories between $\mathcal{T}_1$ and $\mathcal{T}_2$.
The surjectivity is trivial.

REMARK: The existence of quotient affine group $\mathcal{T}$-schemes is assured by 5.14 (ii) [6].

1.8. Lemma
Let $\mathcal{T}$ *be a Tannakian category over* k*, with fundamental group* $\pi(\mathcal{T})$.

(i) If $\mathcal{T}_1, \mathcal{T}_2$ *are two Tannakian subcategories of* $\mathcal{T}$ *such that* $\mathcal{T}_1 \subseteq \mathcal{T}_2$*, then* $H_{\mathcal{T}}(\mathcal{T}_1) \supseteq H_{\mathcal{T}}(\mathcal{T}_2)$.

(ii) If H_1, H_2 *are two normal subgroups of* $\pi(\mathcal{T})$ *such that* $H_1 \subseteq H_2$*, then* $\mathcal{T}(H_1) \supseteq \mathcal{T}(H_2)$*. In particular we have the projection*

$$\pi(\mathcal{T}(H_1)) = \pi(\mathcal{T})/H_1 \longrightarrow \pi(\mathcal{T}(H_2)) = \pi(\mathcal{T})/H_2.$$

1.9. EXAMPLE: Let $\mathcal{T}$ be a Tannakian category over k. The inclusion $i : \mathrm{Vec}(k) \hookrightarrow \mathcal{T}$ of the Tannakian category $\mathrm{Vec}(k)$ of finite-dimensional k-vector spaces in $\mathcal{T}$, defines the faithfully flat morphism of affine group $\mathcal{T}$-schemes

$$I : \pi(\mathcal{T}) \longrightarrow i\mathrm{Spec}\,(k).$$

whose kernel is the whole $\mathcal{T}$-scheme $\pi(\mathcal{T})$: $H_{\mathcal{T}}(\mathrm{Vec}(k)) = \pi(\mathcal{T})$. Hence the functor i identifies the category $\mathrm{Vec}(k)$ with the Tannakian subcategory of objects of $\mathcal{T}$ on which the action (1.3.3) of $\pi(\mathcal{T})$ is trivial (cf. [6] 6.7 (i)).

1.10. Let ω be a fibre functor of the Tannakian category $\mathcal{T}$ over a k-scheme S, namely an exact k-linear $\otimes$-functor from $\mathcal{T}$ to the category of quasi-coherent sheaves over S. It defines a $\otimes$-functor, denoted again ω, from $\mathrm{Ind}\mathcal{T}$ to the category of quasi-coherent sheaves over S. If $\pi(\mathcal{T}) = \mathrm{Sp}(\Lambda)$ we define

$$\omega(\pi(\mathcal{T})) = \mathrm{Spec}(\omega(\Lambda)). \tag{1.10.1}$$

According [7] (8.13.1), the spectrum $\mathrm{Spec}(\omega(\Lambda))$ is the affine group S-scheme $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$ which represents the functor which associates to each S-scheme T, $u : T \longrightarrow S$, the group of automorphisms of $\otimes$-functors of the functor

$$\begin{aligned} \omega_T : \mathcal{T} &\longrightarrow \{\text{locally free scheaves of finite rank over } T\} \\ X &\longmapsto u^*\omega(X). \end{aligned}$$

From the formalism of [6] 5.11, we have the following dictionary:

- to give the affine group $\mathcal{T}$-scheme $\pi(\mathcal{T}) = \mathrm{Sp}(\Lambda)$ is the same thing as to give, for each fibre functor ω over a k-scheme S, the affine group S-scheme $\underline{\mathrm{Aut}}_S^{\otimes}(\omega)$, in a functorial way with respect to ω and in a compatible way with respect to the base changes $S' \longrightarrow S$.
- let $u : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ be a k-linear $\otimes$-functor between Tannakian categories over k. To give the corresponding morphism $U : \pi(\mathcal{T}_2) \longrightarrow u\pi(\mathcal{T}_1)$ of affine group $\mathcal{T}_2$-schemes, is the same thing as to give, for each fibre functor ω of $\mathcal{T}_2$ over a k-scheme S, a morphism of affine group S-schemes $\underline{\mathrm{Aut}}_S^{\otimes}(\omega) \longrightarrow \underline{\mathrm{Aut}}_S^{\otimes}(\omega \circ u)$, in a functorial way with respect to ω.

1.11. **Lemma**

Let $u_1 : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ and $u_2 : \mathcal{T}_2 \longrightarrow \mathcal{T}_3$ be two exact and k-linear $\otimes$-functors between Tannakian categories over k. Denote by $U_1 : \pi(\mathcal{T}_2) \longrightarrow u_1\pi(\mathcal{T}_1)$ and $U_2 : \pi(\mathcal{T}_3) \longrightarrow u_2\pi(\mathcal{T}_2)$ the morphisms of affine group $\mathcal{T}_2$-schemes and $\mathcal{T}_3$-schemes defined respectively by u_1 and u_2. Then the morphism of affine group $\mathcal{T}_3$-schemes corresponding to $u_2 \circ u_1$ is

$$U = u_2U_1 \circ U_2 : \pi(\mathcal{T}_3) \longrightarrow u_2\pi(\mathcal{T}_2) \longrightarrow u_2u_1\pi(\mathcal{T}_1).$$

Moreover,

(i) if $u_2 \circ u_1 \equiv 1_{\mathcal{T}_3}$ then $U : \pi(\mathcal{T}_3) \longrightarrow \mathrm{Sp}\,(1_{\mathcal{T}_3})$,

(ii) if $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3$ and $u_2 \circ u_1 = id$, then $U = id$.

PROOF: The morphism of affine group $\mathcal{T}_2$-schemes $U_1 : \pi(\mathcal{T}_2) \longrightarrow u_1\pi(\mathcal{T}_1)$ furnishes a morphism of affine group $\mathcal{T}_3$-schemes

$$u_2U_1 : u_2\pi(\mathcal{T}_2) \longrightarrow u_2u_1\pi(\mathcal{T}_2)$$

which corresponds to the following systeme of morphisms: for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S, we have a morphism of affine group S-schemes

$$\text{(1.11.1)} \qquad \underline{\mathrm{Aut}}_S^{\otimes}(\omega \circ u_2) \longrightarrow \underline{\mathrm{Aut}}_S^{\otimes}\big((\omega \circ u_2) \circ u_1\big).$$

Denote by $U : \pi(\mathcal{T}_3) \longrightarrow u_2u_1\pi(\mathcal{T}_1)$ the morphism of affine group $\mathcal{T}_3$-schemes corresponding to the functor $u_2 \circ u_1 : \mathcal{T}_1 \longrightarrow \mathcal{T}_3$. To have the morphism U (resp. U_2) of $\mathcal{T}_3$-schemes is the same thing as to have, for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S, a morphism of affine group S-schemes

$$\text{(1.11.2)} \qquad \begin{array}{rl} & \underline{\mathrm{Aut}}_S^{\otimes}(\omega) \longrightarrow \underline{\mathrm{Aut}}_S^{\otimes}\big(\omega \circ (u_2 \circ u_1)\big) \\ \text{(resp.} & \underline{\mathrm{Aut}}_S^{\otimes}(\omega) \longrightarrow \underline{\mathrm{Aut}}_S^{\otimes}\big(\omega \circ u_2\big) \quad) \end{array}$$

Hence, according to (1.11.1) we observe that $U = u_2U_1 \circ U_2$.
The remaining assertions are clear from (1.11.2): in particular, if $u_2 \circ u_1 \equiv 1_{\mathcal{T}_3}$, we have that $\underline{\mathrm{Aut}}_S^{\otimes}\big(\omega_{|\langle 1_{\mathcal{T}_3}\rangle^{\otimes}}\big) = \mathrm{Spec}\,(k)$ for each fibre functor ω of $\mathcal{T}_3$ over a k-scheme S.

2. Some motivic Galois groups.

2.1. Let $MR(k)$ be the category of mixed realizations (for absolute Hodge cycles) over k defined by U. Jannsen in [8] I 2.1. It is a neutral Tannakian category over $\mathbb{Q}$, i.e. a tensor category over $\mathbb{Q}$ with a fibre functor over $\mathrm{Spec}\,(\mathbb{Q})$. Each embedding $\sigma : k \longrightarrow \mathbb{C}$ gives a fibre functor ω_σ of $MR(k)$ over $\mathrm{Spec}\,(\mathbb{Q})$, called "the Hodge realization". If $k = \mathbb{Q}$, there is another fibre functor ω_{dR} of $MR(k)$ over $\mathrm{Spec}\,(\mathbb{Q})$, called "the de Rham realization".

The category of motives over k is the Tannakian subcategory of $MR(k)$ generated (by means of $\oplus$, $\otimes$, dual, subquotients) by those realizations which come from geometry (cf. [6] 1.11).
A motive over k **with integral coefficients** is a motive M whose mixed realization is endowed with an integral structure, i.e. for each embedding $\sigma : k \to \mathbb{C}$, the Hodge realization $\mathrm{T}_\sigma(M)$ of M contains a $\mathbb{Z}$-lattice L_σ and for each prime number ℓ, the ℓ-adic realization $\mathrm{T}_\ell(M)$ of M contains a $\mathrm{Gal}(\overline{k}/k)$-invariant $\mathbb{Z}_\ell$-lattice L_ℓ, such that $I_{\overline{\sigma},\ell}(\mathrm{L}_\sigma) = \mathrm{L}_\ell$ for each ℓ and $\overline{\sigma}$ (cf. [6] 1.23). 1-Motives over k are motives with integral coefficients.

In this paper we are interested in two kinds of motives over k: Artin motives and 1-motives.

2.2. By [5] 6.17 we have a fully faithful functor from the category of 0-dimensional varieties over k to the category $MR(k)$ of mixed realizations over k

$$\begin{aligned}\{0-\text{dim. varieties}/k\} &\longrightarrow MR(k)\\ X &\longmapsto (\mathrm{T}_\sigma(X), \mathrm{T}_{\mathrm{dR}}(X), \mathrm{T}_\ell(X), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})_{\substack{\sigma:k\to\mathbb{C},\ \overline{\sigma}:\overline{k}\to\mathbb{C}\\ \ell \text{ prime number}}}\end{aligned}$$

which associates to each 0-dimensional varietie X its Hodge, de Rham and ℓ-adic realizations and the comparison isomorphisms $I_{\sigma,\mathrm{dR}} : \mathrm{T}_\sigma(X) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathrm{T}_{\mathrm{dR}}(X) \otimes_k \mathbb{C}$ and $I_{\overline{\sigma},\ell} : \mathrm{T}_\sigma(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longrightarrow \mathrm{T}_\ell(X)$. Hence we can identify 0-dimensional varieties with the mixed realizations they define.

The **Tannakian category of Artin motives $\mathcal{T}_0(k)$ over** k is the Tannakian subcategory of $MR(k)$ generated by 0-dimensional varieties over k, i.e. by mixed realizations of 0-dimensional varietes. *$\mathcal{T}_0(k)$ is a neutral Tannakian category over $\mathbb{Q}$ with fibre functors $\{\omega_\sigma\}_{\sigma:k\longrightarrow\mathbb{C}}$ "the Hodge realizations".* The unit object of $\mathcal{T}_0(k)$ is $\mathrm{Spec}\,(k)$.
Through the Hodge realization, $\mathcal{T}_0(k)$ is equivalent to the category of finite-dimensional $\mathbb{Q}$-representations of $\mathrm{Gal}(\overline{k}/k)$, i.e.

$$\begin{aligned}\mathcal{T}_0(k) &\cong \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))\\ X &\longmapsto \mathbb{Q}^{X(\overline{k})}.\end{aligned} \tag{2.2.1}$$

Here and in the following we regard $\mathrm{Gal}(\overline{k}/k)$ as a constant, pro-finite affine group scheme over $\mathbb{Q}$. It is clear that $\mathcal{T}_0(\overline{k})$ is equivalent to the Tannakian category of finite-dimensional $\mathbb{Q}$-vector spaces : in fact, as affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/\overline{k})$ is $\mathrm{Spec}\,(\mathbb{Q})$, and so

$$\mathcal{T}_0(\overline{k}) \cong \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Spec}\,(\mathbb{Q})) \cong \mathrm{Vec}(\mathbb{Q}). \tag{2.2.2}$$

Artin motives are pure motives of weight 0 : the weight filtration W_* on an Artin motive X is $\mathrm{W}_i(X) = X$ for each $i \geq 0$ and $\mathrm{W}_j(X) = 0$ for each $j \leq -1$. If we denote $\mathrm{Gr}_n^{\mathrm{W}} = \mathrm{W}_n/\mathrm{W}_{n-1}$, we have $\mathrm{Gr}_0^{\mathrm{W}}(X) = X$ and $\mathrm{Gr}_i^{\mathrm{W}}(X) = 0$ for each $i \neq 0$.

2.3. A **1-motive** M over k consists of

(a) a group scheme X over k, which is locally for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,

(b) a semi-abelian variety G defined over k, i.e. an extention of an abelian variety A by a torus $Y(1)$, which cocharacter group Y,

(c) a morphism $u : X \longrightarrow G$ of group schemes over k.

We have to think of X as a character group of a torus defined over k, i.e. as a finitely generated $\mathrm{Gal}(\overline{k}/k)$-module. We identify $X(\overline{k})$ with a free $\mathbb{Z}$-module finitely generated. The morphism $u : X \longrightarrow G$ is equivalent to a $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism $u : X(\overline{k}) \longrightarrow G(\overline{k})$.

An **isogeny** between two 1-motives $M_1 = [X_1 \stackrel{u_1}{\longrightarrow} G_1]$ and $M_2 = [X_2 \stackrel{u_2}{\longrightarrow} G_2]$ is a morphism of 1-motives (i.e. a morphism of complexes of commutative group schemes) such that $f_X : X_1 \longrightarrow X_2$ is injective with finite cokernel, and $f_G : G_1 \longrightarrow G_2$ is surjective with finite kernel.

1-motives are mixed motives of niveau ≤ 1: the weight filtration W_* on $M = [X \stackrel{u}{\longrightarrow} G]$ is

$$\begin{aligned} \mathrm{W}_i(M) &= M \quad \text{for each } i \geq 0, \\ \mathrm{W}_{-1}(M) &= [0 \longrightarrow G], \\ \mathrm{W}_{-2}(M) &= [0 \longrightarrow Y(1)], \\ \mathrm{W}_j(M) &= 0 \quad \text{for each } j \leq -3. \end{aligned}$$

In particular, we have $\mathrm{Gr}_0^{\mathrm{W}}(M) = [X \longrightarrow 0], \mathrm{Gr}_{-1}^{\mathrm{W}}(M) = [0 \longrightarrow A]$ and $\mathrm{Gr}_{-2}^{\mathrm{W}}(M) = [0 \longrightarrow Y(1)]$.

2.4. According to [4] 10.1.3 (other [9] 4.2 (i) other for $k = \mathbb{Q}$ [6] 2.3) we have a fully faithful functor from the category of 1-motives over k to the category $MR(k)$ of mixed realizations over k

$$\begin{aligned} \{1 - \text{motives}/k\} &\longrightarrow MR(k) \\ M &\longmapsto (\mathrm{T}_\sigma(M), \mathrm{T}_{\mathrm{dR}}(M), \mathrm{T}_\ell(M), I_{\sigma,\mathrm{dR}}, I_{\overline{\sigma},\ell})_{\substack{\sigma:k\to\mathbb{C},\ \overline{\sigma}:\overline{k}\to\mathbb{C} \\ \ell \text{ prime number}}} \end{aligned}$$

which associates to each 1-motive M its Hodge, de Rham and ℓ-adic realizations and the comparison isomorphisms $I_{\sigma,\mathrm{dR}} : \mathrm{T}_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathrm{T}_{\mathrm{dR}}(M) \otimes_k \mathbb{C}$ and $I_{\overline{\sigma},\ell} : \mathrm{T}_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longrightarrow \mathrm{T}_\ell(M)$. Therefore we can identify 1-motives with the mixed realizations they define.

The **Tannakian category $\mathcal{T}_1(k)$ of 1-motives over** k is the Tannakian subcategory of $MR(k)$ generated by 1-motives, i.e. by mixed realizations of 1-motives: *$\mathcal{T}_1(k)$ is a neutral Tannakian category over $\mathbb{Q}$ with fibre functors $\{\omega_\sigma\}_{\sigma:k\longrightarrow\mathbb{C}}$ "the Hodge realizations".* The unit object of $\mathcal{T}_1(k)$ is the 1-motive $\mathbb{Z}(0) = [\mathbb{Z} \longrightarrow 0]$. For each object M of $\mathcal{T}_1(k)$, we denote by $M^\vee = \underline{\mathrm{Hom}}(M, \mathbb{Z}(0))$ its dual. The Cartier dual of an object M of $\mathcal{T}_1(k)$ is the object

$$M^* = M^\vee \otimes \mathbb{Z}(1) \tag{2.4.1}$$

of $\mathcal{T}_1(k)$. If M is a 1-motive, then M^* is again a 1-motive.

We will denote by $W_{-1}\mathcal{T}_1(k)$ (resp. $\mathrm{Gr}_0^W \mathcal{T}_1(k)$, ...) the Tannakian subcategory of $\mathcal{T}_1(k)$ generated by the mixed realizations of all $W_{-1}M$ (resp. $\mathrm{Gr}_0^W M$, ...) with M a 1-motive.

2.5. If a Tannakian category $\mathcal{T}$ is generated by motives, the fundamental group $\pi(\mathcal{T})$ is called the **motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T})$ of $\mathcal{T}$** : hence, it is the affine group $\mathcal{T}$-scheme Sp(Λ), where Λ is the Hopf algebra of Ind$\mathcal{T}$ satisfying the universal property (1.3.1). If the Tannakian category is generated by only one motive M, i.e. $\mathcal{T} = \langle M \rangle^{\otimes}$, $\pi(\mathcal{T})$ is the **motivic Galois group $\mathcal{G}_{\mathrm{mot}}(M)$ of M:** in this case, it is the affine group $\mathcal{T}$-scheme Sp(Λ), where the Hopf algebra Λ satisfying the universal property (1.3.1) is an object of $\mathcal{T}$. (cf. [7] 6.12).

Since the fundamental group depends only on the Tannakian category, the notion of motivic Galois group is stable under isogeny and duality.

After identifying motives with their mixed realizations and after taking the Tannakian category defined by those mixed realizations, we lose the integral structure (in fact the Tannakian categories we consider are $\mathbb{Q}$-linear). This implies that the motivic Galois group, which depends only on the Tannakian category generated by motives, doesn't "see" the integral structure of motives. Hence, we will work with iso-motives.

2.6. EXAMPLES:

(1) The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))$ of the unit object $\mathbb{Z}(0)$ of $\mathcal{T}_1(k)$ is the affine group $\langle \mathbb{Z}(0) \rangle^{\otimes}$-scheme Sp$(\mathbb{Z}(0))$. For each fibre functor "Hodge realization" ω_σ, we have that $\omega_\sigma\big(\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(0))\big) := \mathrm{Spec}\big(\omega_\sigma(\mathbb{Z}(0))\big) = \mathrm{Spec}(\mathbb{Q})$, which is the Mumford-Tate group of $\mathrm{T}_\sigma(\mathbb{Z}(0))$.

(2) Let $\langle \mathbb{Z}(1) \rangle^{\otimes}$ be the neutral Tannakian category over $\mathbb{Q}$ defined by the k-torus $\mathbb{Z}(1)$. The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathbb{Z}(1))$ of the torus $\mathbb{Z}(1)$ is the affine group $\langle \mathbb{Z}(1) \rangle^{\otimes}$-scheme $\mathcal{G}_m$ defined by the affine $\mathbb{Q}$-scheme $\mathbb{G}_{m/\mathbb{Q}}$ (cf. remark 1.2 (1)). For each fibre functor "Hodge realization" ω_σ, we have that $\omega_\sigma\big(\mathcal{G}_m\big) = \mathbb{G}_{m/\mathbb{Q}}$, which is the Mumford-Tate group of $\mathrm{T}_\sigma(\mathbb{Z}(1))$.

(3) If k is algebraically closed, the motivic Galois group of motives of CM-type over k is the Serre group (cf. [10] 4.8).

2.7. **Lemma-Definition**

The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_0(k))$ of $\mathcal{T}_0(k)$ is the affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$ endowed with its action on itself by inner automorphisms. We denote it by $\mathcal{GAL}(\overline{k}/k)$ In particular, for any fibre functor ω over Spec $(\mathbb{Q})$ *of $\mathcal{T}_0(k)$, the affine group scheme $\omega(\mathcal{GAL}(\overline{k}/k)) = \underline{\mathrm{Aut}}^{\otimes}_{\mathrm{Spec}\,(\mathbb{Q})}(\omega)$ is canonically isomorphic to* $\mathrm{Gal}(\overline{k}/k)$

PROOF: Since $\mathcal{T}_0(k) \cong \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))$, this lemma is an immediate consequence of remark 1.4 (2).

2.8. **Lemma**

(i) The Tannakian subcategory $\mathcal{T}_0(k)$ of $MR(k)$ is equivalent (as tensor category) to the Tannakian subcategory $\mathrm{Gr}_0^W \mathcal{T}_1(k)$.

(ii) We have the following anti-equivalence of tensor categories

$$\begin{aligned} \mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} &\longrightarrow \mathrm{Gr}_{-2}^W \mathcal{T}_1(k) \\ X \otimes \mathbb{Z}(1) &\longmapsto X^{\vee}(1). \end{aligned}$$

PROOF:
(i) It is a consequence of (2.2.1) and of the fact that, as observed in 2.5, in the Tannakian category $MR(k)$ we lose the integral structures.
(ii) According to (i), we can view an object X of $\mathcal{T}_0(k)$ as the character group of a torus T defined over k. The dual $X^\vee$ of X in the Tannakian category $MR(k)$, can be identified with the cocharacter group of T which can be written, according to our notation, as $X^\vee(1)$. The anti-equivalence between the category of character groups and the category of cocharacter groups furnishes the desired anti-equivalence.

2.9. **Corollary**

(i) $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) = \mathcal{GAL}(\overline{k}/k)$,

(ii) $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)) = i_1 \mathcal{GAL}(\overline{k}/k) \times i_2 \mathcal{G}_m$,

where $i_1 : \mathcal{T}_0(k) = \mathcal{T}_0(k) \otimes \mathrm{Vec}(\mathbb{Q}) \longrightarrow \mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$ *and* $i_2 : \langle \mathbb{Z}(1) \rangle^{\otimes} = \mathrm{Vec}(\mathbb{Q}) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes} \longrightarrow \mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$ *identify respectively* $\mathcal{T}_0(k)$ *and* $\langle \mathbb{Z}(1) \rangle^{\otimes}$ *with full subcategories of* $\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}$ *. (In the following, we will avoid the symbols* i_1 *and* i_2*).*

PROOF:
(i) Consequence of (2.8.1).
(ii) From 2.8 (ii) and (2.40.5) [10], we have that

$$\begin{aligned}\mathcal{G}_{\mathrm{mot}}\big(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)\big) =& \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(k) \otimes \langle \mathbb{Z}(1) \rangle^{\otimes}\big) \\ =& i_1 \mathcal{G}_{\mathrm{mot}}\big(\mathcal{T}_0(k)\big) \times i_2 \mathcal{G}_m = i_1 \mathcal{GAL}(\overline{k}/k) \times i_2 \mathcal{G}_m.\end{aligned}$$

2.10. REMARKS:

(1) The motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_0(\overline{k}))$ is the affine group $\mathcal{T}_0(\overline{k})$-scheme $\mathrm{Sp}\,(1_{\mathcal{T}_0(\overline{k})})$ defined by the affine group $\mathbb{Q}$-scheme $\mathrm{Spec}\,(\mathbb{Q})$ (cf. (2.2.2) and 1.4 (2)).

(2) Since the category $\mathrm{Gr}_{-2}^W \mathcal{T}_1(\overline{k})$ is equivalent to the Tannakian category generated by the torus $\mathbb{Z}(1)$, the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(\overline{k}))$ is $\mathcal{G}_m$.

(3) In the category of affine group $\mathcal{T}_0(k)$-schemes, there are two $\mathcal{T}_0(k)$-schemes defined by the Galois group $\mathrm{Gal}(\overline{k}/k)$:

- the affine group $\mathcal{T}_0(k)$-scheme $\mathcal{GAL}(\overline{k}/k)$ which is the affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$ endowed with its action on itself by inner automorphisms. It is the fundamental group of the Tannakian category $\mathcal{T}_0(k)$ of Artin motives.

- the affine group $\mathcal{T}_0(k)$-scheme $\mathrm{Gal}(\overline{k}/k)$ which is the affine group $\mathbb{Q}$-scheme $\mathrm{Gal}(\overline{k}/k)$ endowed with the trivial action of $\mathrm{Gal}(\overline{k}/k)$ (cf. 1.2 (2)). It is a $\mathbb{Q}$-scheme viewed as a $\mathcal{T}_0(k)$-scheme.

Same remark for the affine group $\langle \mathbb{Z}(1) \rangle^{\otimes}$-schemes $\mathcal{G}_m$ and $\mathbb{G}_m$.

2.11. The weight filtration W_* on objects of $\mathcal{T}_1(k)$ induces an increasing filtration, always denoted by W_*, on the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of $\mathcal{T}_1(k)$ (cf.

[11] Chapitre IV §2). We describe this filtration through the action (1.3.3) of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ on the generators of $\mathcal{T}_1(k)$. For each 1-motive M over k,

$W_0(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$

$W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-1}(M),$
$(g - id)W_{-1}(M) \subseteq W_{-2}(M), (g - id)W_{-2}(M) = 0\},$

$W_{-2}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \{g \in \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \mid (g - id)M \subseteq W_{-2}(M),$
$(g - id)W_{-1}(M) = 0\},$

$W_{-3}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = 0.$

The step $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ is an affine unipotent group sub-$\mathcal{T}_1(k)$-scheme of the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. According to 2.9 and to the motivic analogue of [3] §2.2, $\mathrm{Gr}_0^W(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ acts through $\mathcal{GAL}(\overline{k}/k)$ on $\mathrm{Gr}_0^W\mathcal{T}_1(k)$, and through $\mathcal{GAL}(\overline{k}/k)\times\mathcal{G}_m$ on $\mathrm{Gr}_{-2}^W\mathcal{T}_1(k)$. Moreover, its image in the automorphisms group of $\mathrm{Gr}_{-1}^W\mathcal{T}_1(k)$ is the motivic Galois group of the abelian part $\mathrm{Gr}_{-1}^W\mathcal{T}_1(k)$. Therefore, $\mathrm{Gr}_0^W(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ is an affine reductive group sub-$\mathcal{T}_1(k)$-scheme of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ and $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ is the unipotent radical of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$.

2.12. Consider the base extension functor

$$\begin{aligned} E : \mathcal{T}_1(k) &\longrightarrow \mathcal{T}_1(\overline{k}) \\ M &\longmapsto M \otimes_k \overline{k}. \end{aligned} \tag{2.12.1}$$

According to (2.2.2), the kernel of E is $\mathcal{T}_0(k)$, i.e. we have the exact sequence

$$0 \longrightarrow \mathcal{T}_0(k) \longrightarrow \mathcal{T}_1(k) \xrightarrow{E} \mathcal{T}_1(\overline{k}). \tag{2.12.2}$$

If M is an object of $\mathcal{T}_1(\overline{k})$, it can be written as a subquotient of $M' \otimes_k \overline{k}$ for some object M' of $\mathcal{T}_1(k)$: in fact, for M' we can take the restriction of scalars $\mathrm{Res}_{k'/k}M_0$ with M_0 a model of M over a finite extension k' of k. By 1.5, this means that the corresponding morphism of affine group $\mathcal{T}_1(\overline{k})$-schemes

$$e : \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) \hookrightarrow E\,\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \tag{2.12.3}$$

is a closed immersion.

Finally, denote by

$$\begin{aligned} \mathrm{Res}_{\overline{k}/k} : \mathcal{T}_1(\overline{k}) &\longrightarrow \mathcal{T}_1(k) \\ M &\longmapsto \mathrm{Res}_{\overline{k}/k}(M) \end{aligned} \tag{2.12.4}$$

the functor "restriction of scalars".

3. Tannakian subcategories of $\mathcal{T}_1(k)$.

3.1. Consider the following diagram of inclusions of Tannakian categories

$$\begin{tikzcd}
\mathrm{Gr}_0^W \mathcal{T}_1(k) = \mathcal{T}_0(k) \arrow[dr, "I_0"] & & \\
\mathrm{Gr}_{-1}^W \mathcal{T}_1(k) \arrow[r, "I_{-1}"] & \mathrm{Gr}_*^W \mathcal{T}_1(k) \arrow[r, "I"] & \mathcal{T}_1(k) \\
\mathrm{Gr}_{-2}^W \mathcal{T}_1(k) \arrow[ur, "I_{-2}"] & &
\end{tikzcd}$$

and denote by $\mathcal{I}_j : \mathrm{Gr}_j^W \mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k)$ the inclusion $I \circ I_j$ for $j = 0, -1, -2$.
According to 1.5, to this diagram corresponds the diagram of faithfully flat morphisms of affine group $\mathcal{T}_1(k)$-schemes

$$\begin{tikzcd}
& & \mathcal{I}_0 \mathcal{GAL}(\overline{k}/k) \\
\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \arrow[r, "i"] & I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \arrow[ur, "Ii_o"] \arrow[r, "Ii_{-1}"] \arrow[dr, "Ii_{-2}"] & \mathcal{I}_{-1} \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)) \\
& & \mathcal{I}_{-2}(\mathcal{GAL}(\overline{k}/k) \times \mathcal{G}_m)
\end{tikzcd}$$

where $Ii_j : I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \longrightarrow \mathcal{I}_j \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_j^W \mathcal{T}_1(k))$ is the morphism of affine group $\mathcal{T}_1(k)$-schemes defined by the morphism of affine group $\mathrm{Gr}_*^W \mathcal{T}_1(k)$-schemes $i_j : \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \longrightarrow I_j \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_j^W \mathcal{T}_1(k))$ corresponding to the inclusion I_j.
For $j = 0, -1, -2$, denote by

$$\iota_j : \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \longrightarrow \mathcal{I}_j \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_j^W \mathcal{T}_1(k))$$

the faithfully flat morphism of affine group $\mathcal{T}_1(k)$-schemes corresponding to the inclusion $\mathcal{I}_j$. In particular, by 1.11 we have that $\iota_j = Ii_j \circ i$.

Always according to 1.5, the functor "take the graduated" $\mathrm{Gr}_*^W : \mathcal{T}_1(k) \longrightarrow \mathrm{Gr}_*^W \mathcal{T}_1(k)$ corresponds to the closed immersions of affine group $\mathrm{Gr}_*^W \mathcal{T}_1(k)$-schemes

$$\mathrm{gr}_*^W : \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \longrightarrow \mathrm{Gr}_*^W \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)),$$

which identifies the motivic Galois group of $\mathrm{Gr}_*^W \mathcal{T}_1(k)$ with the quotient Gr_0^W of the motivic Galois group of $\mathcal{T}_1(k)$.
Consider now the following commutative diagram of inclusion of Tannakian categories

$$
\begin{array}{ccccccc}
\mathrm{Gr}_0^W \mathcal{T}_1(k) = \mathcal{T}_0(k) & \stackrel{J_3}{\to} & & W_0/W_{-2}\mathcal{T}_1(k) & & & \\
 & \stackrel{J_4}{\nearrow} & & & & \stackrel{J_1}{\searrow} & \\
\mathrm{Gr}_{-1}^W \mathcal{T}_1(k) & & & \stackrel{\mathcal{I}_{-1}}{\to} & & & \mathcal{T}_1(k) \\
 & \stackrel{J_5}{\searrow} & & & & \stackrel{J_2}{\nearrow} & \\
\mathrm{Gr}_{-2}^W \mathcal{T}_1(k) & \stackrel{J_6}{\to} & & W_{-1}\mathcal{T}_1(k) & & &
\end{array}
$$

where $J_1 \circ J_3 = \mathcal{I}_0$ and $J_2 \circ J_6 = \mathcal{I}_{-2}$. By 1.5, to this diagram corresponds the commutative diagram of faithfully flat morphisms of affine group $\mathcal{T}_1(k)$-schemes

$$
\begin{array}{ccccccc}
 & & J_1\mathcal{G}_{\mathrm{mot}}(W_0/W_{-2}\mathcal{T}_1(k)) & & \stackrel{J_1j_3}{\to} & & \mathcal{I}_0\mathcal{GAL}(\overline{k}/k) \\
 & \stackrel{j_1}{\nearrow} & & & \stackrel{J_1j_4}{\searrow} & & \\
\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & & & \stackrel{\iota_{-1}}{\to} & & & \mathcal{I}_{-1}\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)) \\
 & \stackrel{j_2}{\searrow} & & & \stackrel{J_2j_5}{\nearrow} & & \\
 & & J_2\mathcal{G}_{\mathrm{mot}}(W_{-1}\mathcal{T}_1(k)) & & \stackrel{J_2j_6}{\to} & & \mathcal{I}_{-2}(\mathcal{GAL}(\overline{k}/k) \times \mathcal{G}_m)
\end{array}
\tag{3.1.1}
$$

where by 1.11 we have that $\iota_0 = J_1 j_3 \circ j_1$ and $\iota_{-2} = J_2 j_6 \circ j_2$.

3.2. According to 1.7, the Tannakian subcategories $\mathcal{T}_0(k), \mathrm{Gr}_{-1}^W \mathcal{T}_1(k)$, $\mathrm{Gr}_{-2}^W \mathcal{T}_1(k), W_0/W_{-2}\mathcal{T}_1(k), W_{-1}\mathcal{T}_1(k)$ are characterized by the following affine group sub-$\mathcal{T}_1(k)$-schemes of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$.

$$
\begin{aligned}
H_{\mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) &= \ker \big[\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{\iota_0} \mathcal{I}_0\mathcal{GAL}(\overline{k}/k)\big], \\
H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)) &= \ker \big[\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{\iota_{-1}} \mathcal{I}_{-1}\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k))\big], \\
H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)) &= \ker \big[\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{\iota_{-2}} \mathcal{I}_{-2}(\mathcal{GAL}(\overline{k}/k) \times \mathcal{G}_m)\big], \\
H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k)) &= \ker \big[\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{j_1} J_1\mathcal{G}_{\mathrm{mot}}(W_0/W_{-2}\mathcal{T}_1(k))\big], \\
H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) &= \ker \big[\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{j_2} J_2\mathcal{G}_{\mathrm{mot}}(W_{-1}\mathcal{T}_1(k))\big].
\end{aligned}
$$

These affine group sub-$\mathcal{T}_1(k)$-schemes are the motivic generalization of the algebraic $\mathbb{Q}$-groups introduced in [1] §2. Because of the commutativity of the diagram (3.1.1), we have the inclusions

$$
\begin{aligned}
H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) &\subseteq H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)) \cap H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)), \\
H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k)) &\subseteq H_{\mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) \cap H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)).
\end{aligned}
$$

By lemma 2.8, the Tannakian category $\mathrm{Gr}_0^W \mathcal{T}_1(k)$ of Artin motives is a Tannakian subcategory of $\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)$ and therefore according to 1.8 we have that

$$
H_{\mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) \supseteq H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)). \tag{3.2.1}
$$

Moreover the Cartier duality furnishes the anti-equivalence of tensor categories

$$\begin{aligned} W_0/W_{-2}\mathcal{T}_1(k) &\longrightarrow W_{-1}\mathcal{T}_1(k) \\ M &\longmapsto M^*. \end{aligned}$$

We denote both these categories by $\widetilde{W}\mathcal{T}_1(k)$. In particular we have that

$$H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k)) = H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) = H_{\mathcal{T}_1(k)}(\widetilde{W}\mathcal{T}_1(k)). \tag{3.2.2}$$

With these notation, we can state the motivic generalization of lemma [1] 2.2 to all motives of niveau ≤ 1.

3.3. **Lemma**
(i) $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = \cap_{i=-1,-2} H_{\mathcal{T}_1(k)}(\mathrm{Gr}_i^W \mathcal{T}_1(k))$,
(ii) $W_{-2}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))) = H_{\mathcal{T}_1(k)}(\widetilde{W}\mathcal{T}_1(k))$.

PROOF: Let M be a 1-motive over k. We remark that

$$\begin{aligned}
g \in H_{\mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) &\Longleftrightarrow g_{|\mathrm{Gr}_0^W(M)} = id \Longleftrightarrow (g-id)M \subseteq W_{-1}(M) \\
g \in H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-1}^W \mathcal{T}_1(k)) &\Longleftrightarrow g_{|\mathrm{Gr}_{-1}^W(M)} = id \Longleftrightarrow (g-id)W_{-1}(M) \subseteq W_{-2}(M) \\
g \in H_{\mathcal{T}_1(k)}(\mathrm{Gr}_{-2}^W \mathcal{T}_1(k)) &\Longleftrightarrow g_{|\mathrm{Gr}_{-2}^W(M)} = id \Longleftrightarrow (g-id)W_{-2}(M) = 0 \\
g \in H_{\mathcal{T}_1(k)}(W_0/W_{-2}\mathcal{T}_1(k)) &\Longleftrightarrow g_{|W_0/W_{-2}(M)} = id \Longleftrightarrow (g-id)M \subseteq W_{-2}(M) \\
g \in H_{\mathcal{T}_1(k)}(W_{-1}\mathcal{T}_1(k)) &\Longleftrightarrow g_{|W_{-1}(M)} = id \Longleftrightarrow (g-id)W_{-1}(M) = 0.
\end{aligned}$$

The result now follows from (3.2.1), (3.2.2) and from the definitions of $W_{-1}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$ and of $W_{-2}(\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)))$.

3.4. **Theorem**
We have the following diagram of morphisms of affine group sub-$\mathcal{T}_1(k)$-schemes

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) & \to & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \stackrel{\iota_0}{\to} & \mathcal{I}_0\mathcal{GAL}(\overline{k}/k) & \to & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \to & \mathrm{Res}_{\overline{k}/k}H_{\mathcal{T}_1(\overline{k})}(\langle\mathbb{Z}(1)\rangle^{\otimes}) & \to & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \stackrel{\iota_{-2}}{\to} & \mathcal{I}_{-2}(\mathcal{GAL}(\overline{k}/k)\times\mathcal{G}_m) & \to & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \to & W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \to & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \stackrel{i}{\to} & I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(k)) & \to & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \to & W_{-2}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \to & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \stackrel{j_1}{\to} & J_1\mathcal{G}_{\mathrm{mot}}(\widetilde{W}\mathcal{T}_1(k)) & \to & 0
\end{array}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.

PROOF: We will prove the exactness of the four horizontal short sequences applying theorem 1.7 to the following Tannakian subcategories of $\mathcal{T}_1(k)$: $\mathcal{T}_0(k), \widetilde{W}\mathcal{T}_1(k)$, $\mathrm{Gr}_{-2}^W\mathcal{T}_1(k)$ and $\mathrm{Gr}_*^W\mathcal{T}_1(k)$.
By [8] 2.20 (e) the composite functor $\mathrm{Res}_{\overline{k}/k} \circ E$ is the multiplication by $[\overline{k} : k]$. Therefore, since we work modulo isogenies, according (2.12.3) the affine group $\mathcal{T}_1(k)$-scheme $\mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k}))$ is a sub-$\mathcal{T}_1(k)$-scheme of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$. Now consider the inclusion $\mathcal{I}_0 : \mathrm{Gr}_0^W\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k)$ (cf. 3.1). Since Artin motives are the kernel of the base extension functor (cf. (2.12.2)), the objects of $\mathcal{T}_0(k)$ are exactly those objects of $\mathcal{T}_1(k)$ on which the sub-$\mathcal{T}_1(k)$-scheme $\mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k}))$ of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ acts trivially. Hence by 1.7, we have the exact sequence of affine group $\mathcal{T}_1(k)$-schemes

$$0 \longrightarrow \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{\iota_0} \mathcal{I}_0\mathcal{GAL}(\overline{k}/k) \longrightarrow 0.$$

As in 3.1 denote by $\mathcal{I}_{-2}$ the inclusion of the Tannakian subcategory $\mathrm{Gr}_{-2}^W\mathcal{T}_1(k)$ in the Tannakian category $\mathcal{T}_1(k)$. As observed in 2.10 (2), $\mathrm{Gr}_{-2}^W\mathcal{T}_1(\overline{k})$ is equivalent as tensor category to the Tannakian subcategory $\langle\mathbb{Z}(1)\rangle^\otimes$ of $\mathcal{T}_1(\overline{k})$ generated by the k-torus $\mathbb{Z}(1)$. Hence the objects of $\mathrm{Gr}_{-2}^W\mathcal{T}_1(k)$ are exactly those objects of $\mathcal{T}_1(k)$ on which, after extension of scalars, the sub-$\mathcal{T}_1(\overline{k})$-scheme $H_{\mathcal{T}_1(\overline{k})}(\langle\mathbb{Z}(1)\rangle^\otimes)$ of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k}))$ acts trivially. Applying 1.7, we have the exact sequence of affine group $\mathcal{T}_1(k)$-schemes

$$0 \longrightarrow \mathrm{Res}_{\overline{k}/k}H_{\mathcal{T}_1(\overline{k})}(\langle\mathbb{Z}(1)\rangle^\otimes) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{\iota_{-2}} \mathcal{I}_{-2}(\mathcal{GAL}(\overline{k}/k) \times \mathcal{G}_m) \longrightarrow 0$$

Consider now the inclusion $i : \mathrm{Gr}_*^W\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k)$ (cf. 3.1). Since the objects of $\mathrm{Gr}_*^W\mathcal{T}_1(k)$ are exactly those objects of $\mathcal{T}_1(k)$ on which the unipotent radical $W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ acts trivially, the affine group $\mathcal{T}_1(k)$-scheme characterizing $\mathrm{Gr}_*^W\mathcal{T}_1(k)$ is

$$H_{\mathcal{T}_1(k)}(\mathrm{Gr}_*^W\mathcal{T}_1(k)) = W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$$

(cf. [6] 6.7 (ii)). In particular, we have the exact sequence of affine group $\mathcal{T}_1(k)$-schemes

$$0 \longrightarrow W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{i} I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(k)) \longrightarrow 0$$

and as expected the motivic Galois group $\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(k))$ is isomorph to $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))/W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$.
As in 3.1 let $J_1 : \widetilde{W}\mathcal{T}_1(k) \longrightarrow \mathcal{T}_1(k)$ be the inclusion of the Tannakian subcategory $\widetilde{W}\mathcal{T}_1(k)$ in the Tannakian category $\mathcal{T}_1(k)$. According to 3.3 (ii) and 1.7 we have the exact sequence of affine group $\mathcal{T}_1(k)$-schemes

$$0 \longrightarrow W_{-2}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \xrightarrow{j_1} J_1\mathcal{G}_{\mathrm{mot}}(\widetilde{W}\mathcal{T}_1(k)) \longrightarrow 0.$$

Finally, in order to prove that the left vertical arrows are inclusions and that the right vertical arrows are surjections, it is enough to apply the lemma 1.8.

3.5. As corollary, we get the motivic version of [5] II 6.23 (a), (c) and [8] 4.7 (c), (e). But before to state this corollary, we recall some facts:

If $F_1, F_2 : \mathcal{T}_1(\overline{k}) \longrightarrow \mathcal{T}_1(\overline{k})$ are two functors, we define $\underline{\mathrm{Hom}}^{\otimes}(F_1, F_2)$ to be the functor which associates to each $\mathcal{T}_1(\overline{k})$-scheme $\mathrm{Sp}(B)$, the set of morphisms of $\otimes$-functors from $(F_1)_{\mathrm{Sp}(B)} : X \longmapsto F_1(X) \otimes B$ to $(F_2)_{\mathrm{Sp}(B)} : X \longmapsto F_2(X) \otimes B$ ($(F_1)_{\mathrm{Sp}(B)}$ and $(F_2)_{\mathrm{Sp}(B)}$ are $\otimes$-functors from $\mathcal{T}_1(\overline{k})$ to the category of modules over $\mathrm{Sp}(B)$).

Moreover, each element τ of $\mathrm{Gal}(\overline{k}/k)$ defines a functor

$$\tau : \mathcal{T}_1(\overline{k}) \longrightarrow \mathcal{T}_1(\overline{k})$$

in the following way: since as observed in 2.12, the category $\mathcal{T}_1(\overline{k})$ is generated by motives of the form $E(M)$ with $M \in \mathcal{T}_1(k)$, it is enough to define $\tau E(M)$. We put $\tau E(M) = M \otimes_k \tau\overline{k}$.

3.6. **Corollary**

(i) We have the following diagram of morphisms of affine group sub-$\mathcal{T}_1(k)$-schemes in which all the short sequences are exact:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & W_{-1}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & & & & \\
 & & & & \downarrow & & & & \\
0 & \rightarrow & \mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(\overline{k})) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) & \xrightarrow{\iota_0} & \mathcal{I}_0\mathcal{GAL}(\overline{k}/k) & \rightarrow & 0 \\
 & & \downarrow & & i\downarrow & & \| & & \\
0 & \rightarrow & I\mathrm{Res}_{\overline{k}/k}\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(\overline{k})) & \rightarrow & I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(k)) & \xrightarrow{Ii_0} & II_0\mathcal{GAL}(\overline{k}/k) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

(ii) The morphism $I\mathrm{gr}_*^W : I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W\mathcal{T}_1(k)) \longrightarrow \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ of affine group $\mathcal{T}_1(k)$-schemes is a section of i.
(iii) For any $\tau \in \mathcal{GAL}(\overline{k}/k)$, $\iota_0^{-1}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, regarding Id and $\tau \circ \mathrm{Id}$ as functors on $\mathcal{T}_1(\overline{k})$. In an analogous way, $Ii_0^{-1}(\tau) = \underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, regarding Id and $\mathrm{Id} \circ \tau$ as functors on $\mathrm{Gr}_*^W\mathcal{T}_1(\overline{k})$.

PROOF: (i) We have only to prove the exactness of the second horizontal short sequence. In order to do this, we apply theorem 1.7 to the Tannakian category $\mathrm{Gr}_0^W\mathcal{T}_1(k)$ viewed this time as subcategory of $\mathrm{Gr}_*^W\mathcal{T}_1(k)$. Consider the inclusion

$I_0 : \mathrm{Gr}_0^W \mathcal{T}_1(k) \longrightarrow \mathrm{Gr}_*^W \mathcal{T}_1(k)$ (cf 3.1). By (2.12.2), Artin motives are exactly the kernel of the base extension functor and therefore

$$H_{\mathrm{Gr}_*^W \mathcal{T}_1(k)}(\mathrm{Gr}_0^W \mathcal{T}_1(k)) = \mathrm{Res}_{\overline{k}/k} \mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(\overline{k})).$$

(ii) Since

$$I \circ \mathrm{Gr}_*^W = id : \mathcal{T}_1(k) \stackrel{\mathrm{Gr}_*^W}{\longrightarrow} \mathrm{Gr}_*^W \mathcal{T}_1(k) \stackrel{I}{\longrightarrow} \mathcal{T}_1(k),$$

from 1.11 we have that

$$I\mathrm{gr}_*^W \circ i = id : \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)) \stackrel{i}{\longrightarrow} I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W \mathcal{T}_1(k)) \stackrel{I\mathrm{gr}_*^W}{\longrightarrow} \mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k)).$$

(iii) By [7] 8.11, the fundamental group $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))$ represents the functor $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})$ which associates to each $\mathcal{T}_1(k)$-scheme $\mathrm{Sp}(B)$ the group of automorphisms of $\otimes$-functors of the functor

$$\begin{aligned} \mathrm{Id}_{\mathrm{Sp}(B)} : \mathcal{T}_1(k) &\longrightarrow \{\text{modules over } \mathrm{Sp}(B)\} \\ X &\longmapsto X \otimes B. \end{aligned}$$

Hence if g is an element of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))(\mathrm{Sp}\, B) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})(\mathrm{Sp}\, B)$, for each pair of objects M and N of $\mathcal{T}_1(k)$ and for each morphism $f : M \longrightarrow N$ of $\mathcal{T}_1(k)$, we have the commutative diagram

$$\begin{array}{ccc} M \otimes B & \stackrel{g_M}{\longrightarrow} & M \otimes B \\ f \otimes id_B \downarrow & & \downarrow f \otimes id_B \\ N \otimes B & \stackrel{g_N}{\longrightarrow} & N \otimes B. \end{array}$$

Let M and N be two objects of $\mathcal{T}_1(k)$. Since $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}(E(M), E(N))$ is an object of $\mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))$, it can be regarded as an Artin motive over k. Moreover, the elements of $\mathrm{Hom}_{\mathcal{T}_1(k)}(M, N)$ are exactly the elements of $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}(E(M), E(N))$ which are invariant under the action of $\mathrm{Gal}(\overline{k}/k)$, i.e.

$$\mathrm{Hom}_{\mathcal{T}_1(k)}(M, N) = \big(\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}(E(M), E(N))\big)^{\mathrm{Gal}(\overline{k}/k)}.$$

Let g be an element of $\mathcal{G}_{\mathrm{mot}}(\mathcal{T}_1(k))(\mathrm{Sp}\, B) = \underline{\mathrm{Aut}}^{\otimes}(\mathrm{Id})(\mathrm{Sp}\, B)$, and let $\iota_0(g) = \tau \in \mathrm{Gal}(\overline{k}/k)$. This means that g acts via τ on $\mathrm{Hom}_{\mathcal{T}_1(\overline{k})}(E(M), E(N))$. Then for any morphism $h : E(M) \longrightarrow E(N)$ of $\mathcal{T}_1(\overline{k})$, we have the commutative diagram

$$\begin{array}{ccc} E(M) \otimes B & \stackrel{E(g_M)}{\longrightarrow} & E(M) \otimes B \\ h \otimes id_B \downarrow & & \downarrow \tau h \otimes id_B \\ E(N) \otimes B & \stackrel{E(g_N)}{\longrightarrow} & E(N) \otimes B. \end{array} \tag{3.6.1}$$

Since M and N are defined over k, $E(M)$ and $E(N)$ are respectively isomorph to $\tau E(M)$ and $\tau E(N)$ and therefore the upper line of (3.6.1) defines a morphism

$$E(M) \otimes B \longrightarrow \tau E(M) \otimes B$$

which is functorial in $E(M)$ and B, and which is compatible with tensor products. Moreover we have already observed in 2.12 that the Tannakian category $\mathcal{T}_1(\overline{k})$ is generated by motives of the form $E(M)$ with $M \in \mathcal{T}_1(k)$. We can then conclude that g defines an element of $\underline{\mathrm{Hom}}^{\otimes}(\mathrm{Id}, \tau \circ \mathrm{Id})$, regarding Id and $\tau \circ \mathrm{Id}$ as functors on $\mathcal{T}_1(\overline{k})$.

4. Specialization to a 1-motive.

4.1. In this section we need a more symmetric description of 1-motives: consider the 7-uplet $(X, Y^\vee, A, A^*, v, v^*, \psi)$ where

- X and $Y^\vee$ are two group k-schemes, which are locally for the étale topology, constant group schemes defined by a finitely generated free $\mathbb{Z}$-module;
- A and A^* are two abelian varieties defined over k, dual to each other;
- $v : X \longrightarrow A$ and $v^* : Y^\vee \longrightarrow A^*$ are two morphisms of group schemes over k; and
- ψ is a trivialization of the pull-back $(v, v^*)^* \mathcal{P}_A$ by (v, v^*) of the Poincaré biextension $\mathcal{P}_A$ of (A, A^*).

By [4] (10.2.14), to have the data $(X, Y^\vee, A, A^*, v, v^*, \psi)$ is equivalent to have the 1-motive $M = [X \xrightarrow{u} G]$, where G is the semi-abelian variety defined by the homomorphism v^* and u is the lifting of v determined by the trivialization ψ.

4.2. Let $M = (X, Y^\vee, A, A^*, v, v^*, \psi)$ be a 1-motive defined over k. It is not true that the Tannakian category generated by $W_0/W_{-2}(M)$ is equivalent to the Tannakian category generated by $W_{-1}(M)$. Hence the lemma 3.3. must be modified in the following way

$$\begin{aligned} W_{-1}(\mathcal{G}_{\mathrm{mot}}(M)) &= \cap_{i=-1,-2} H_{\mathcal{T}_1(k)}(\mathrm{Gr}_i^W M), \\ W_{-2}(\mathcal{G}_{\mathrm{mot}}(M)) &= H_{\mathcal{T}_1(k)}(W_0/W_{-2}(M)) \cap H_{\mathcal{T}_1(k)}(W_{-1}(M)). \end{aligned} \tag{4.2.1}$$

Theorem 3.4 becomes: We have the following diagram of morphisms of affine group sub-$\langle M \rangle^{\otimes}$-schemes

(4.2.2)

$$\begin{array}{ccccccccc}
0 & \rightarrow & W_{-1}\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(M) & \stackrel{i}{\rightarrow} & I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W M) & \rightarrow & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \rightarrow & W_{-2}\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & \mathcal{G}_{\mathrm{mot}}(M) & \stackrel{j}{\rightarrow} & J\mathcal{G}_{\mathrm{mot}}(W_0/W_{-2}(M)+W_{-1}(M)) & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & W_{-2}\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & W_{-1}\mathcal{G}_{\mathrm{mot}}(M) & \rightarrow & W_{-1}J\mathcal{G}_{\mathrm{mot}}(W_0/W_{-2}(M)+W_{-1}(M)) & \rightarrow & 0
\end{array}$$

where the morphisms i and j correspond to the inclusions $I : \langle \mathrm{Gr}_*^W M\rangle^{\otimes} \longrightarrow \langle M\rangle^{\otimes}$ and $J : \langle W_0/W_{-2}(M) + W_{-1}(M)\rangle^{\otimes} \longrightarrow \langle M\rangle^{\otimes}$, where all horizontal short sequences are exact and where all vertical arrows are inclusions, except for $J\mathcal{G}_{\mathrm{mot}}(W_0/W_{-2}(M) + W_{-1}(M)) \longrightarrow I\mathcal{G}_{\mathrm{mot}}(\mathrm{Gr}_*^W M)$ which is surjective.

4.3. In [2], it is proved that if $M = (X, Y^\vee, A, A^*, v, v^*, \psi)$ is a 1-motive over k, the unipotent radical of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the semi-abelian variety defined by the adjoint action of the Lie algebra $(\mathrm{Gr}_*^W(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)), [\,,\,])$ on itself. The abelian variety B and the torus $Z(1)$ underlying this semi-abelian variety can be computed explicitly. We recall here briefly their construction:
The motive $E = W_{-1}(\underline{\mathrm{End}}(\mathrm{Gr}_*^W M))$ is a split 1-motive whose non trivial components are the abelian variety $A \otimes X^\vee + A^* \otimes Y$ and the torus $X^\vee \otimes Y(1)$. Moreover E is endowed of a Lie crochet $[\,,\,]$ whose non trivial component

$$(A \otimes X^\vee + A^* \otimes Y) \otimes (A \otimes X^\vee + A^* \otimes Y) \longrightarrow X^\vee \otimes Y(1).$$

is defined through the morphism $A \otimes A^* \longrightarrow \mathbb{Z}(1)$ associated by [2] (1.3.1) to the Poincaré biextension $\mathcal{P}$ of (A, A^*) by $\mathbb{Z}(1)$. According to corollary [2] 2.7, this Lie crochet corresponds to a $\Sigma - X^\vee \otimes Y(1)$-torsor $\mathcal{B}$ living over $A \otimes X^\vee + A^* \otimes Y$. As proved in [2] 3.3, the 1-motives $\mathrm{Gr}_*^W M$ and $\mathrm{Gr}_*^W M^\vee$ are Lie $(E, [\,,\,])-$modules. In particular, E acts on the components $\mathrm{Gr}_0^W M$ and $\mathrm{Gr}_0^W M^\vee$ in the following way:

$$\begin{aligned} \alpha :&(X^\vee \otimes A) \otimes X \longrightarrow A \\ \beta :&(A^* \otimes Y) \otimes Y^\vee \longrightarrow A^* \\ \gamma :&(X^\vee \otimes Y(1)) \otimes X \longrightarrow Y(1) \end{aligned} \tag{4.3.1}$$

These morphisms are projections defined through the evaluation maps $ev_{X^\vee} : X^\vee \otimes X \longrightarrow \mathbb{Z}(0)$ and $ev_{Y^\vee} : Y^\vee \otimes Y \longrightarrow \mathbb{Z}(0)$ of the Tanakian category $\mathcal{T}_1(k)$ (cf. [7] (2.1.2)).
Thank to the morphisms $\delta_{X^\vee} : \mathbb{Z}(0) \longrightarrow X \otimes X^\vee$ and $ev_X : X \otimes X^\vee \longrightarrow \mathbb{Z}(0)$, which characterize the notion of duality in a Tannakian category (cf. [7] (2.1.2)), to have the morphisms $v : X \longrightarrow A$ and $v^* : Y^\vee \longrightarrow A^*$ is equivalent to have a k-rational point $b = (b_1, b_2)$ of the abelian variety $A \otimes X^\vee + A^* \otimes Y$.

- Let B be the smallest abelian sub-variety (modulo isogeny) of $X^\vee \otimes A + A^* \otimes Y$ containing the point $b = (b_1, b_2) \in X^\vee \otimes A(k) \times A^* \otimes Y(k)$.

The restriction $i^*\mathcal{B}$ of the $\Sigma - X^\vee \otimes Y(1)$-torsor $\mathcal{B}$ by the inclusion $i : B \longrightarrow X^\vee \otimes A \times A^* \otimes Y$ is a $\Sigma - X^\vee \otimes Y(1)$-torsor over B.

- Denote by Z_1 the smallest $\mathrm{Gal}(\overline{k}/k)$-module of $X^\vee \otimes Y$ such that the torus $Z_1(1)$ that it defines, contains the image of the Lie crochet $[\,,\,] : B \otimes B \longrightarrow X^\vee \otimes Y(1)$.

The direct image $p_* i^*\mathcal{B}$ of the $\Sigma - X^\vee \otimes Y(1)$-torsor $i^*\mathcal{B}$ by the projection $p : X^\vee \otimes Y(1) \longrightarrow (X^\vee \otimes Y/Z_1)(1)$ is a trivial $\Sigma - (X^\vee \otimes Y/Z_1)(1)$-torsor over B, i.e.

$p_*i^*\mathcal{B} = B \times (X^\vee \otimes Y/Z_1)(1)$. We denote by $\pi : p_*i^*\mathcal{B} \longrightarrow (X^\vee \otimes Y/Z_1)(1)$ the canonical projection.
By [2] 3.6, to have the lifting $u : X \longrightarrow G$ of the morphism $v : X \longrightarrow A$ is equivalent to have a point $\widetilde{b}$ in the fibre of $\mathcal{B}$ over b. We denote again by $\widetilde{b}$ the points of $i^*\mathcal{B}$ and of $p_*i^*\mathcal{B}$ over the point b of B.

- Let Z be the smallest sub-$\mathrm{Gal}(\overline{k}/k)$-module of $X^\vee \otimes Y$, containing Z_1 and such that the sub-torus $(Z/Z_1)(1)$ of $(X^\vee \otimes Y/Z_1)(1)$ contains $\pi(\widetilde{b})$. If we put $Z_2 = Z/Z_1$, we have that $Z(1) = Z_1(1) \times Z_2(1)$.

With these notation, the Lie algebra $(\mathrm{Gr}_*^W(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)), [\,,\,])$ is the Lie algebra $(B+Z(1), [\,,\,])$ and the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ of the Lie algebra of $\mathcal{G}_{\mathrm{mot}}(M)$ is the extension of the abelian variety B by the torus $Z(1)$ defined by the adjoint action of $(B + Z(1), [\,,\,])$ on itself.

4.4. **Proposition**
The derived group of the unipotent radical $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ *of the Lie algebra of* $\mathcal{G}_{\mathrm{mot}}(M)$ *is the torus* $Z_1(1)$.

PROOF: This result is a consequence of the motivic version of the structural lemma [1] 1.4. In fact, let $g_1 = (P, Q, \vec{v})$ and $g_2 = (R, S, \vec{w})$ be two elements of $B + Z(1)$, with $P, R \in B \cap X^\vee \otimes A(k)$, $Q, S \in B \cap A^* \otimes Y(k)$, and $\vec{v}, \vec{w} \in Z(1)(k)$. We have

$$g_1 \circ g_2 = (P + R, Q + S, \vec{v} + \vec{w} + \Upsilon(P, Q, R, S) \tag{4.4.1}$$

where Υ is a $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism from $(X^\vee \otimes A + A^* \otimes Y)(\overline{k}) \otimes (X^\vee \otimes A + A^* \otimes Y)(\overline{k})$ to $X^\vee \otimes Y(\overline{k})$. In order to determine Υ, we have to understand how $g_1 \circ g_2$ acts on the 1-motive M. The 1-motives $M/W_{-2}M$ and $W_{-1}M$ generate Tannakian subcategories of $\langle M/W_{-2}M + W_{-1}M\rangle^\otimes$ and therefore by 1.5 we have the surjective morphisms

(4.4.2)
$$pr_1 : W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M + W_{-1}M\rangle^\otimes) \longrightarrow W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M\rangle^\otimes)$$
$$pr_2 : W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M + W_{-1}M\rangle^\otimes) \longrightarrow W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle W_{-1}M\rangle^\otimes).$$

Since by [2] 3.10 $\mathrm{Gr}_{-1}^W\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the abelian variety B and $W_{-2}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the torus $Z(1)$, according the third short exact sequence of (4.2.2) we get that $W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M + W_{-1}M\rangle^\otimes)$ is the abelian variety B. It follows that explicitly the morphisms (4.4.2) are the projections

$$pr_1 : B \longrightarrow B \cap X^\vee \otimes A$$
$$pr_2 : B \longrightarrow B \cap A^* \otimes Y.$$

Let $\pi : W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)) \longrightarrow W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(\langle M/W_{-2}M + W_{-1}M\rangle^{\otimes}))$ be the surjective morphism coming from the faithfully flat morphism of the third short exact sequence of (4.2.2). By definition of $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M/W_{-2}M))$ we have that

$$\begin{gathered}\big(pr_1(\pi\, g_1) - id\big)\, W_0/W_{-2}(M) \subseteq A,\\ \big(pr_1(\pi\, g_1) - id\big)\, A = 0.\end{gathered}$$

Hence modulo the canonical isomorphism $\underline{\mathrm{Hom}}(X;A) \cong X^{\vee} \otimes A$ which allows us to identify $pr_1(\pi\, g_1) - id$ with $P \in B \cap X^{\vee} \otimes A(k)$, we obtain that

$$P : \mathrm{Gr}_0^W(M) \longrightarrow A. \tag{4.4.3}$$

In an analogous way, by definition of $W_{-1}(\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(W_{-1}M))$ we observe that

$$\begin{gathered}\big(pr_2(\pi\, g_2) - id\big)\, W_{-1}M \subseteq T,\\ \big(pr_2(\pi\, g_2) - id\big)\, T = 0.\end{gathered} \tag{4.4.4}$$

Since the Cartier dual of $W_{-1}(M)$ is the 1-motive $M^*/W_{-2}M^*$, $pr_2(\pi\, g_2)$ acts on a contravariant way on $M^*/W_{-2}M^*$, and therefore we have that

$$\begin{gathered}\big(pr_2(\pi\, g_2)' - id\big)\, M^*/W_{-2}M^* \subseteq A^*,\\ \big(pr_2(\pi\, g_2)' - id\big)\, A^* = 0,\end{gathered}$$

where the symbol $'$ denote the contravariant action. Consequently, modulo the canonical isomorphism $\underline{\mathrm{Hom}}(Y^{\vee};A^*) \cong A^* \otimes Y$ which allows us to identify $pr_2(\pi\, g_2)' - id$ with $S \in B \cap A^* \otimes Y$, we have by (4.4.4) that

$$-S : A \longrightarrow Y(1). \tag{4.4.5}$$

Again modulo the canonical isomorphism $\underline{\mathrm{Hom}}(X;Y(1)) \cong X^{\vee} \otimes Y(1)$, from (4.4.3) and (4.4.5) we get

$$-[P;S] : \mathrm{Gr}_0^W(M) \xrightarrow{P} A \xrightarrow{-S} Y(1).$$

It follows that $\Upsilon(P,Q,R,S) = -[P;S]$ and using (4.4.1) we can conclude that

$$g_1 \circ g_2 - g_2 \circ g_1 = (0, 0, -[P;S] + [R;Q])$$

which is an element of $Z_1(1)$ by definition.

4.5. Let $\{e_i\}_i$ and $\{f_j^*\}_j$ be basis of $X(\overline{k})$ and $Y^{\vee}(\overline{k})$ respectively. Choose a point P of $B \cap X^{\vee} \otimes A(k)$ and a point Q of $B \cap A^* \otimes Y(k)$ such that the abelian sub-variety they generate in $X^{\vee} \otimes A + A^* \otimes Y$, is isogeneous to B. Denote by $\overline{v} : X(\overline{k}) \longrightarrow A(\overline{k})$ et $\overline{v}^* : Y^{\vee}(\overline{k}) \longrightarrow A^*(\overline{k})$ the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphisms defined by

$$\overline{v}(e_i) = \alpha(P, e_i),$$
$$\overline{v}^*(f_j^*) = \beta(Q, f_j^*).$$

Moreover choose a point $\vec{q} = (q_1, \dots, q_{\mathrm{rg}\, Z_2})$ of $Z_2(1)(k)$ such that the points $q_1, \dots, q_{\mathrm{rg}\, Z_2}$ are multiplicative independent.
Let $\Gamma : Z(1)(\overline{k}) \otimes X \otimes Y^{\vee}(\overline{k}) \longrightarrow \mathbb{Z}(1)(\overline{k})$ be the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism obtained from the maps $\gamma : (X^{\vee} \otimes Y(1)) \otimes X \longrightarrow Y(1)$ and $ev_Y : Y \otimes Y^{\vee} \longrightarrow \mathbb{Z}(0)$, and denote by $\overline{\psi} : X \otimes Y^{\vee}(\overline{k}) \longrightarrow \mathbb{Z}(1)(\overline{k})$ the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism defined by

$$\overline{\psi}(e_i, f_j^*) = \Gamma([P, Q], \vec{q}, e_i, f_j^*). \tag{4.5.1}$$

4.6. **Lemma**
With the above notation, the Tannakian category $\langle M \rangle^{\otimes}$ is equivalent to the Tannakian category generated by the 1-motives $(e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j})$, where $\overline{v}_i, \overline{v}_j^*$ and $\overline{\psi}_{i,j}$ are the $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphisms obtained restricting respectively $\overline{v}, \overline{v}^*$ and $\overline{\psi}$ to $e_i\mathbb{Z}$ and $f_j^*\mathbb{Z}$:

$$\begin{aligned}
\overline{v}_i : e_i\mathbb{Z} \longrightarrow A, &\qquad e_i \longmapsto \alpha(P, e_i),\\
\overline{v}_j^* : f_j^*\mathbb{Z} \longrightarrow A^*, &\qquad f_j^* \longmapsto \alpha^*(Q, f_j^*),\\
\overline{\psi}_{i,j} : e_i\mathbb{Z} \times f_j^*\mathbb{Z} \longrightarrow \mathcal{P}_{|e_i\mathbb{Z} \times f_j^*\mathbb{Z}}, &\qquad (e_i, f_j^*) \longmapsto \Gamma([P, Q], \vec{q}, e_i, f_j^*).
\end{aligned}$$

PROOF: According to the proof of theorem 3.8 [2], the homomorphisms $\overline{v}, \overline{v}^*$ and $\overline{\psi}$ define a 1-motive $(X, Y^{\vee}, A, A^*, \overline{v}, \overline{v}^*, \overline{\psi})$ which generates the same Tannakian category as M. In order to conclude we apply theorem [1] 1.7 to the 1-motives $(X, Y^{\vee}, A, A^*, \overline{v}, \overline{v}^*, \overline{\psi})$ and $(e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j})$.

4.7. Consider the 1-motives

$$\begin{aligned}
M^{tab} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, 0, 0, 0, 0, \overline{\psi}_{i,j}^{ab})\\
M^{a} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, 0)\\
M^{nab} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}^{nab})\\
M^{ab} &= \oplus_{i,j}\, (e_i\mathbb{Z}, f_j^*\mathbb{Z}, A, A^*, \overline{v}_i, \overline{v}_j^*, \overline{\psi}_{i,j}^{ab})
\end{aligned}$$

where

$$\begin{aligned}
\overline{\psi}_{i,j}^{nab}(e_i, f_j^*) &= \Gamma([P, Q], \vec{1}, e_i, f_j^*)\\
\overline{\psi}_{i,j}^{ab}(e_i, f_j^*) &= \Gamma(\vec{1}, \vec{q}, e_i, f_j^*)
\end{aligned} \tag{4.7.1}$$

According to lemma 4.6, the 1-motives M^{tab}, M^a, M^{ab} and M^{nab} belong to the Tannakian category $\langle M \rangle^{\otimes}$ generated by M.

4.8. **Lemma**
The Tannakian category generated by M is equivalent to the Tannakian category generated by the 1-motive $M^{tab} \oplus M^{nab}$. Moreover the 1-motives M^{ab} and $M^a \oplus M^{tab}$ generate the same Tannakian category.

PROOF: Since through the projection $p : X^{\vee} \otimes Y(1) \longrightarrow (X^{\vee} \otimes Y/Z_1)(1)$ the $\Sigma - (X^{\vee} \otimes Y)(1)$-torsor $i^*\mathcal{B}$ becomes a trivial torsor, i.e. $p_* i^* \mathcal{B} = B \times (X^{\vee} \otimes Y/Z_1)(1)$, confronting (4.5.1) and (4.7.1), we observe that to have the trivialization $\overline{\psi}^{ab}_{i,j}$ and $\overline{\psi}^{nab}_{i,j}$ is the same thing as to have the trivialization $\overline{\psi}_{i,j}$. Hence by lemma 4.6, we have that the 1-motives M and $M^{tab} \oplus M^{nab}$ generate the same Tannakian category.
Always because of the fact that the $\Sigma - (X^{\vee} \otimes Y/Z_1)(1)$-torsor $p_* i^* \mathcal{B}$ is trivial, we observe that the trivialization $\overline{\psi}^{ab}_{i,j}$ is independent of the abelian part of the 1-motive M, i.e. it is independent of $\overline{v}_i, \overline{v}^*_j$ Therefore, we can conclude that the Tannakian category generated by 1-motive M^{ab} is equivalent to the Tannakain category generated by the 1-motive $M^a \oplus M^{tab}$.

4.9. **Theorem**
The Tannakian category generated by M^{ab} is the biggest Tannakian subcategory of $\langle M \rangle^{\otimes}$ whose motivic Galois group is commutative. We have the following diagram of morphisms of affine group sub-$\langle M \rangle^{\otimes}$-schemes

$$\begin{array}{ccccccccc}
0 & \to & Z_2(1) & \to & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \stackrel{i^{nab}}{\to} & I^{nab}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{nab}) & \to & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \to & Z(1) & \to & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \stackrel{i^{a}}{\to} & I^{a}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{a}) & \to & 0 \\
 & & \uparrow & & || & & \uparrow & & \\
0 & \to & Z_1(1) & \to & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \stackrel{i^{ab}}{\to} & I^{ab}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{ab}) & \to & 0 \\
 & & \downarrow & & || & & \downarrow & & \\
0 & \to & B+Z_1(1) & \to & \mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M) & \stackrel{i^{tab}}{\to} & I^{tab}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M^{tab}) & \to & 0
\end{array}$$

where all horizontal short sequences are exact and where the vertical arrows on the left are inclusions and those on the right are surjections.

PROOF: By [2] 3.10 we know that $\mathrm{Gr}^W_*(W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M))$ is the Lie algebra $(B + Z(1), [\,,\,])$. Now, from the definition of the 1-motives M^{nab}, M^{ab} and M^{tab}, and from (4.7.1) we observe that the torus $Z_2(1)$ acts trivially on M^{nab}, that the torus $Z_1(1)$ acts trivially on M^{ab} and that the split 1-motive $B + Z_1(1)$ acts trivially on M^{tab}. In other words we have that

$$\begin{aligned}
&\mathrm{Lie}\, H_{\langle M\rangle^{\otimes}}(\langle M^{nab}\rangle^{\otimes}) = Z_2(1) \\
&\mathrm{Lie}\, H_{\langle M\rangle^{\otimes}}(\langle M^{ab}\rangle^{\otimes}) = Z_1(1) \\
&\mathrm{Lie}\, H_{\langle M\rangle^{\otimes}}(\langle M^{tab}\rangle^{\otimes}) = B + Z_1(1)
\end{aligned}$$

and therefore we get the first, the second and the fourth short exact sequence. Recall that by [2] 3.10 the Lie algebra $W_{-2}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is the torus $Z(1)$. Moreover by construction, the 1-motive without toric part M^a generates the same Tannakian category as the 1-motive $W_0/W_{-2}M + W_{-1}M$. Hence thanks to (4.2.1) the obtain the second horizontal short exact sequence. (It is possible to prove the exactness of the second short sequence computing the Lie algebra $\mathrm{Lie}\, H_{\langle M\rangle^{\otimes}}(\langle M^a\rangle^{\otimes})$.)

According to lemma 4.8, the 1-motives M^a and M^{tab} generate Tannakian subcategories of the Tannakian category $\langle M^{ab}\rangle^{\otimes}$. By construction the 1-motive M^a generates a Tannakian subcategory of the Tannakian category M^{nab}. Hence in order to prove that the left vertical arrows are inclusions and that the right vertical arrows are surjections, it is enough to apply the lemma 1.8.

The third exact sequence of the above diagram implies that the motivic Galois group of M^{ab} is isomorph to the quotient

$$\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)/Z_1(1)$$

But according proposition 4.4, $Z_1(1)$ is the derived group of $\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ and hence we can conclude that M^{ab} generate the biggest Tannakian subcategory of $\langle M\rangle^{\otimes}$ whose motivic Galois group is commutative.

4.10. REMARK: Among the non degenerate 1-motives, the 1-motive M^{nab} is the one which generates the biggest Tannakian subcategory of $\langle M\rangle^{\otimes}$, whose motivic Galois group is non commutative. (A 1-motive is said to be non degenerate if the dimension of the Lie algebra $W_{-1}\mathrm{Lie}\,\mathcal{G}_{\mathrm{mot}}(M)$ is maximal (cf. [1] 2.3)).

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